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A Müntz Space Having No Complement

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Let Λ be a set of positive numbers for which $\sum_{\lambda \in \Lambda} 1/\lambda < \infty$ and consider the closed span S_Λ of the monomials $1, x^\lambda, \lambda \in \Lambda$. By the celebrated theorem of Müntz, S_Λ is *not* all of $C[0, 1]$, and by a later theorem of Erdös, S_Λ consists only of "power" series $a_0 + \sum_\Lambda a_\lambda x^\lambda$ convergent on $[0, 1)$.

Recently, it was asked whether these S_Λ always have a complement in $C[0, 1]$, or, equivalently, whether there is always a bounded projection of $C[0, 1]$ onto S_Λ. Indeed, it was noticed that this is the case for Λ the powers of 2, but we shall prove that it is *not* always so.

The construction of our example is based on the observation that any projection of $C[0, 1]$ onto the span of $1, x^N, x^{2N},..., x^{kN}$ has norm $> c \log k$. Thus, we can force a *large* projection even though there is a *small* sum of reciprocals. Using this fact allows us to construct our example by a kind of "condensation of singularities." The details are as follows:

LEMMA 1. *There is a $c > 0$ such that any projection of $C[0, 1]$ onto the span of $1, x^N, x^{2N},..., x^{kN}$ has norm $> c \log k$.*

Proof. If P is such a projection and ϕ denotes the map $\phi: f(x) \to f(x^N)$ then $\phi^{-1}P\phi$ is a projection onto the span of $1, x, x^2,..., x^k$. Writing $x = (\cos \theta + 1)/2$, $0 \leqslant \theta \leqslant \pi$, then we see we have a projection of $C[0, \pi]$ onto $\operatorname{span}\{1, \cos \theta,..., \cos k\theta\}$ and it is known that the smallest such projection is the Fourier projection F. Since $\|F\| \sim (4/\pi^2) \log k$ it follows that $\|\phi^{-1}P\phi\| > c \log k$, $c > 0$. Finally, since ϕ is an isometry we see that $\|P\| = \|\phi^{-1}P\phi\|$ and the lemma is established.

We now need a result which says that a polynomial in x is very poorly approximable by a polynomial in a very high power of x. This is

LEMMA 2. *Suppose $p(x)$ and $q(x)$ are polynomials of degree at most m, and that $q(0) = 0$, then $\| p(x) + q(x^n)\| \geqslant (1 - 2m^2(\log n + 2)/n) \| p(x)\|$.*

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Proof. We may normalize so that

$$\| p(x) + q(x^n) \| = 1. \tag{1}$$

We may also assume $\| p \| \geqslant 1$ or all is trivial, so we write

$$\| p(x) \| = A \geqslant 1, \tag{2}$$

and we deduce from (1), that

$$\| q(x) \| \leqslant A + 1 \leqslant 2A. \tag{3}$$

By the standard derivative bounds, we then obtain

$$\| p'(x) \| \leqslant 2m^2 A, \tag{4}$$

$$\| q'(x) \| \leqslant 4m^2 A. \tag{5}$$

From (5) it follows that $|q(x)| \leqslant 4m^2 Ax$, so that

$$|q(x^n)| \leqslant 4m^2 A \left(1 - \frac{\log n}{n}\right)^n \leqslant \frac{4m^2 A}{n} \qquad \text{on} \quad \left[0, 1 - \frac{\log n}{n}\right], \tag{6}$$

and so, again by (1), we have

$$| p(x) | \leqslant 1 + \frac{4m^2}{n} A \qquad \text{on} \left[0, 1 - \frac{\log n}{n}\right]. \tag{7}$$

On the rest of the unit interval we use (4) to get

$$| p(x) | \leqslant 1 + \frac{4m^2}{n} A + 2m^2 A \frac{\log n}{n} \qquad \text{on all of } [0, 1]. \tag{8}$$

But of course this means that

$$A \leqslant 1 + \frac{4m^2}{n} A + 2m^2 A \frac{\log n}{n}, \tag{9}$$

which is exactly the required result.

We now define:

A *fast* sequence of integers N_i is one, where

$$\frac{N_i}{\log N_i + 2} \geqslant 2N_{i-1}^2 i^2 (i-1)^2, \qquad 1 < N_1 < N_2 \cdots, \tag{10}$$

and, by repeated use of our Lemma 2; we derive

LEMMA 3. *Let N_i be a fast sequence (see* (10)), *write* $N_0 = 0$, *assume* $\deg P_i(x) \leqslant i$ *for* $i \geqslant 0$, $P_i(0) = 0$ *for* $i > 0$, *and call* $S_j = \sum_{i=0}^{j} P_i(x^{N_i})$. *Then we have*

$$\|S_j\| \leqslant 2\,\|S_k\| \qquad \textit{for} \quad k \geqslant j \geqslant 0.$$

Proof. We use induction on k to prove even more, that

$$\|S_k\| \geqslant \frac{k+1}{2k}\,\|S_j\|. \tag{11}$$

The induction begins at $k = j$, where the result is trivial, and we may assume it to hold for $k - 1$, which is to say,

$$\|S_{k-1}\| \geqslant \frac{k}{2(k-1)}\,\|S_j\|. \tag{12}$$

We now apply Lemma 2 to the case of

$$p(x) = S_{k-1}, \qquad q(x) = P_k(x), \qquad n = N_k, \qquad m = N_{k-1}(k-1).$$

Clearly $\deg p(x) \leqslant m$, but also $\deg q(x) \leqslant k \leqslant 2(k-1) \leqslant N_{k-1}(k-1) = m$, and so the hypotheses of Lemma 2 are satisfied. Also $2m^2(\log n + 2)/n = 2N_{k-1}^2(k-1)^2(\log N_k + 2)/N_k \geqslant 1/k^2$ by (10) and so the lemma gives

$$\|S_k\| \geqslant \left(1 - \frac{1}{k^2}\right)\|S_{k-1}\|. \tag{13}$$

Combining (12) and (13) and noting that $(1 - (1/k^2))(k/(k-1)) = (k+1)/k$ does indeed give (11), and so the induction is complete.

We now wish to extend this result to the case $k = \infty$, but this is not quite trivial and some convergence hypothesis is needed. We have, namely,

LEMMA 4. *Under the same hypotheses as in Lemma* 3 *plus the assumption that* $\sum_{i=1}^{\infty} p_i(x^{N_i})$ *converges uniformly on every subinterval* $[0, \rho]$, $\rho < 1$, *we have* $\|S_j\| \leqslant 2\,\|S_\infty\|$.

Proof. Fix a positive $\rho < 1$ and a positive ε. By the uniform convergence we can choose an $M > j$ so that $|S_M| \leqslant (1 + \varepsilon)\,|S_\infty|$ on $[0, \rho]$ which gives

$$\|S_M(\rho x) \leqslant \|S_\infty\| + \varepsilon. \tag{14}$$

Applying Lemma 3 therefore gives

$$\|S_j(\rho x)\| \leqslant 2\,\|S_M(\rho x)\| \leqslant 2\,\|S_\infty\| + 2\varepsilon, \tag{15}$$

and we may now let $\rho \to 1$ and then $\varepsilon \to 0$. (Note that we did not have to assume $\|S_\infty\| < \infty$ since the result is trivial when $\|S_\infty\| = \infty$.)

Having completed these preliminaries it is an easy task to display our example. We simply choose any fast sequence N_i and then take $\Lambda = \bigcup_{i=1}^{\infty} \{N_i, 2N_i, ..., iN_i\}$. To prove that this has the desired property we note that every $f(x) \in S_\Lambda$ has an Erdös expansion $\sum_{i=0}^{\infty} P_i(x^{N_i})$ satisfying the hypotheses of Lemma 4. Therefore, by that lemma, and in the language of that lemma, we have $\|S_j\| \leqslant 2\|f\|$, $\|S_{j-1}\| \leqslant 2\|f\|$, $\|S_0\| \leqslant 2\|f\|$. This means that the map, T_j, taking each $f \in S_\Lambda$ into its $S_j - S_{j-1} + S_0$ has norm bounded by 6. The map T_j, however, is a projection from S_Λ onto $\operatorname{span}\{1, x^{N_j}, x^{2N_j}, ..., x^{jN_j}\}$.

Thus if there were a bounded projection P of $C[0,1]$ onto S_Λ then T_jP would be a projection of $C[0,1]$ onto $\operatorname{span}\{1, x^{N_j}, ..., x^{jN_j}\}$. The bound $\|T_jP\| \leqslant 6\|P\|$ would, therefore, give a contradiction to Lemma 1! Q.E.D.